VOLUME 81 SEPARATE No. 690

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

MAY, 1955

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

STRESSES IN ELASTIC PLATES OVER FLEXIBLE SUBGRADES

by Eric Reissner

ENGINEERING MECHANICS DIVISION

{Discussion open until September 1, 1955}

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort by the Society to deliver technical data direct from the author to the reader with the greatest possible speed. To this end, it has had none of the usual editing required in more formal publication procedures.

Readers are invited to submit discussion applying to current papers. For this paper the final date on which a discussion should reach the Manager of Technical Publications appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings — Separate No. 290). For free copies of this Separate—describing style, content, and format—address the Manager, Technical Publications, ASCE.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

STRESSES IN ELASTIC PLATES OVER FLEXIBLE SUBGRADES

Eric Reissner*

ABSTRACT

Comparative studies are made of the effect flexible surface layers have on load carrying capacity of the combination of surface layer and of subgrade. The analysis is based on the representation of the subgrade by a system of springs or, equivalently, by a heavy liquid.

Simple approximate formulas are obtained for maximum tension stress in the surface layer and maximum deflection of surface layer for loads distributed uniformly in circular areas. An important bound is established for the range of applicability of these formulas and this bound also roughly determines the range of usefulness of the surface layer.

It is shown that in the range of usefulness of the surface layer reasonable non-uniformity of load distribution, and built-in tensions of plausible magnitude are of secondary importance.

It is further shown that non-linear effects due to non-infinitesimal magnitude of deflections are negligible as long as deflections of the surface layer are no larger than about one half the thickness of the surface layer.

INTRODUCTION

The present report has as its object a study of various theoretical questions which arise in connection with the load carrying capacity of foundations consisting of a subgrade strengthened through the presence of a relatively thin surface layer with better strength characteristics than the subgrade itself. The surface layer may be a mat, or a concrete layer, or the subsoil itself, improved down to a certain depth by means of various additives.

Two basic questions confront the investigator at the outset of studies of this kind. They are

(1) what to assume in regard to the nature of the behaviour of the subgrade,

(2) what to assume in regard to the nature of the behaviour of the flexible surface layer.

It appears that in all previous work of this kind the subgrade has been considered as obeying either Hooke's Law of elasticity or it has been assumed that the subgrade behaves like a system of parallel tension-compression springs. The latter assumptic ı is equivalent to the consideration of the subgrade as a heavy liquid and is therefore often referred to as the liquid-subgrade assumption.

It may be considered certain that either of the above two assumptions is at best an approximation to the actual behaviour of the subgrade under load.

*Prof., Dept. of Mathematics, Massachusetts Inst. of Technology, Cambridge, Mass.

Past experience indicates however that the approximation is quite good and corresponds fairly well to reality for subsoils of normal consistencies and loads of such distribution as permit the return to more or less original conditions upon removal of the loads. It is clearly one of the functions of the flexible surface layer to result in a distribution of load on the subsoil permitting the existence of this condition for larger total loads than in the absence of the surface layer.

Regarding the nature of the behaviour of the flexible surface layer it is necessary to carry out somewhat more specific considerations than in the case of the subgrade. We may distinguish, in reference to the action of a flexible surface the following types of action:

1) bending action
2) tension-surface action
3) transverse shear action.

Actions (1) and (3) are always present, although often one of them, generally (3), is negligible compared to the other. Action (2) may or may not be present.

The manner in which the flexible surface may be represented for the purpose of analysis depends essentially on the length ratio, diameter of load surface to thickness of surface layer. For values of this ratio beginning at about unity it is possible to consider the surface layer as a "thin plate" in a manner which will be explained later on, rather than in the more detailed fashion of a three-dimensional continuum.

Nature of results obtained.

The quantities which are determined in what follows are primarily the following two quantities:

1) the maximum value of the pressure on the subgrade at the interface of subgrade and surface layer,

2) the maximum value of the tensile stress in the surface layer.

Determination of these two quantities is carried out in their dependence on what are believed to be the essential parameters which enter into the problem

1) elasticity of the surface layer.
2) thickness of surface layer
3) diameter of load area
4) amount of total load
5) deformation characteristics of the subgrade.

In the course of the present analysis it is found that, subject to specific restrictions, which it is believed are generally met with in practice, certain formulas which were previously used can indeed be justified on wider grounds than was heretofore possible. While some of these restrictions may have been implicit in the earlier work, at least one of them, presently to be formulated, has previously been disregarded explicitly.

Upon due consideration the following decision has been made in regard to the present analyisis. Since the object is the comparative study of surface layers endowed with a variety of properties and since previous analyses have shown fairly close analogies between the "elastic" and the "liquid" subgrade behaviour it was felt justified to base the present work on just one assumption concerning subgrade behaviour. As this one assumption it was natural to choose the simpler one, namely the assumption of a liquid subgrade, characterized by a foundation constant k.

The following four special problems pertaining to the various possible properties of the surface layer have been solved.

(1) Infinite elastic plate carrying a load distributed uniformly over a finite circular area. The plate is treated by conventional plate theory. The results for this case have been given previously by Pickett and co-workers (reference 1) and the present calculations confirm the formulas of these authors. The present report adds to the results of reference 1 a requirement that these formulas not be used beyond a certain value of one of the basic parameters. For values of the parameter within the recommended range it is found possible to simplify the formulas of reference 1 to bring out more clearly the relative significance of thickness changes and elastic modulus changes for the surface layers.

(2) Infinite elastic plate carrying a load distributed parabolically over a finite circular area. Since the actual nature of the load intensity distribution over the assumed circular area is not in actuality known it appears appropriate to investigate the differences in plate stress and foundation pressure intensity due to changes in load distribution, for fixed total load. The relevant calculations are carried out for one family of significant load distributions. It is found that up to a certain point the results for problem 1 and the present problem 2 are the same for practical purposes. Significant discrepancies appear at about the same values of the basic parameter which were found critical for problem 1. This fact lends additional support to the restrictions advised here in connection with the use of the analytical formulas for the plate with uniform load distribution. The formulas for the present problem 2 have been specifically obtained for this report and it is possible that they have not been previously published anywhere.

(3) Infinite elastic plate subjected to uniform tension in plane of plate and carrying uniformly distributed lateral load. In view of the fact that the installation of the flexible surface may build into it a state of pretension (witness the occurrence of tensile cracks in the chemically treated surfaces, due to shrinkage) it seemed advisable to investigate the influence of built-in tension on foundation pressure and plate bending stresses. An explicit solution was obtained for this problem and it is believed that this solution is new. It is found that for the amount of tension possible in chemically treated surface layers the effect of this tension is minor. However, the analytical solution obtained should be of value beyond leading to recognition of this fact since it permits a numerical analysis of similar problems in a unified manner, beginning with the plate without built-in tension all the way up to the case where this tension is the dominating factor and the flexural resistance of the plate is negligible.

(4) Non-linear effects. It is known that in most problems concerning transverse deflections of flat plates the relations between loads and deflections are linear as long as the deflections remain less than about one half of the plate thickness. Since plates on a flexible subgrade apparently had not heretofore been checked in regard to this property such an analysis is given as part of the present report. It is found that here too non-linearity becomes significant only for displacements of half the plate thickness or more. In many or most problems pertaining to flexible surfaces on flexible subgrades this means that the ordinary linear, small-deflection theory is adequate. The present report goes somewhat beyond establishing this result. It also includes a formulation of the non-linear problem which is particularly convenient for the actual calculation of non-linear corrections to the linear theory in the event that such calculations might become desirable.

In order to facilitate the use of this report the main body of it is restricted to the statement of the various problems, the statement of the results and to

some discussion of these results. Detailed derivations are incorporated in the Appendix.

LIST OF SYMBOLS

w = transverse plate deflection in inches
k = modulus of subgrade reaction in pounds per cubic inch
E = modulus of elasticity of surface layer in psi
μ = Poisson's ratio of surface layer material
h = plate thickness in inches
r = radial distance measured from center of load surface
a = radius of circular load surface in inches
p = load intensity function in pounds per square inch
D = plate stiffness factor, $D = Eh^3/12(1 - \nu^2)$
∇^2 = Laplace operator, $\nabla^2 f = d^2f/dr^2 + r^{-1}df/dr$
σ_F = foundation pressure
σ_B = extreme fiber bending stresses in plate
J = symbol for Bessel functions of first kind
Y = symbol for Bessel functions of second kind
H = symbol for Hankel functions
$\lambda = \sqrt{i}\,\rho$
$\rho = \sqrt[4]{k/D}$
$\mu = \lambda a$
u)
v)
U) = symbols for real and imaginary part of certain Bessel functions
V) defined in references 3 and 6
C = pressure coefficient, $C = \sigma_F(0)/p_{average}$
η = parameter in parabolic load distribution
$C = C_F + (1 - \eta)C_F^*$
C_B = parameter defined through equation (11')
σ = hydrostatic two-dimensional tensile stress
N = σh, stress resultant
F = a stress function occurring in non-linear theory
ξ = dimensionless coordinate, $\xi = r/a$

INFINITE ELASTIC PLATE CARRYING A LOAD DISTRIBUTED UNIFORMLY OVER A FINITE CIRCULAR AREA

The differential equation for this problem is

$$D\,\nabla^2\nabla^2 w + kw = p \tag{1}$$

For the present problem the load intensity function p is given by

$$p = \begin{cases} p_0, & r \leq a \\ 0, & a < r \end{cases} \tag{2}$$

where a is the radius of the circular load surface.

To be determined are in particular

(1) the foundation pressure distribution σ_F given by

$$\sigma_F = kw, \tag{3}$$

(2) the bending stresses in the outer most fibers of the plate, in radial and circumferential direction. These are given by

$$\sigma_{rB} = \pm \frac{1}{2} \frac{Eh}{1 - \nu^2} \left[\frac{d^2w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right] \tag{4a}$$

$$\sigma_{\theta B} = \pm \frac{1}{2} \frac{Eh}{1 - \nu^2} \left[\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2w}{dr^2} \right] \tag{4b}$$

Equations (3) and (4) hold independently of the form of the pressure intensity function and will be used also in subsequent sections of this report.

The solution of equations (1) and (2) is of the following form

$$w = \begin{cases} \frac{p_0}{k} \left[1 + \frac{\pi}{2} \quad \mathrm{Im} \left\{ \mu H_1^{(1)}(\mu) J_0(\lambda r) \right\} \right] & , \ r \leq a \\ \frac{p_0}{k} \left[\frac{\pi}{2} \ \mathrm{Im} \left\{ \mu J_1(\mu) H_0^{(1)}(\lambda r) \right\} \right] & , \ a < r \end{cases} \tag{5}$$

where

$$\lambda = \sqrt{i} \ \sqrt[4]{\frac{k}{D}} = \frac{1 + i}{\sqrt{2}} \sqrt[4]{\frac{k}{D}}$$

and

$$\mu = \lambda a$$

and where J_1, $H_0(1)$ and $H_1(1)$ are Bessel and Hankel functions in the usual notation.

For the evaluation of (5) it is to be noted that $\sqrt{i} = e^{i\pi/4}$.

Write further

$$\lambda = \rho e^{i\pi/4}, \quad \rho = \sqrt[4]{k/D}.$$

Then, in the notation of reference 3

$$J_{0,1}(\lambda r) = J_{0,1}(\rho r e^{i\pi/4}) = u_{0,1}(\rho r, \tfrac{\pi}{4}) + i v_{0,1}(\rho r, \tfrac{\pi}{4})$$

$$Y_{0,1}(\lambda r) = Y_{0,1}(\rho r e^{i\pi/4}) = U_0(\rho r, \tfrac{\pi}{4}) + i V_0(\rho r, \tfrac{\pi}{4})$$

and

$$H^{(1)} = J + i Y = [u - V] + i [v + U]$$

Since all these functions, with subscripts zero and one, are tabulated in references 3 and 6 it is quite simple to plot the displacement profile w(r) as given by (5) in its dependence on r.

It is plausible, and may be confirmed by computation, that w is largest at the origin r = 0. Therewith the foundation pressure σ_F as given by (3) is also largest for r = 0. We shall first discuss the value of the maximum foundation pressure $\sigma_F(0)$. From (3) and (5),

$$\sigma_F(0) = p_0 \left[1 + (\pi/2) \ \mathrm{Im}\left\{ \mu H_1^{(1)}(\mu) \right\} \right] \qquad (6)$$

The result of equation (6) is also given in reference 1, in the first and second columns in Table 6 for pressure coefficients C. Allowing for differences in notation the pressure coefficient C is the ratio $\sigma_F(0)/p_0$,

$$C = 1 + \frac{\pi}{2} \mathrm{Im}\left\{ \mu H^{(1)}(\mu) \right\},$$

$$(6')$$

$$\mu = \lambda a = \sqrt{i} \ \rho a = \sqrt{i} \ a \sqrt[4]{k/D} = \sqrt{i} \ a/l$$

The results expressed in equations (6) may alternately be expressed in terms of the so-called Kelvin functions, which are real and imaginary parts of various types of Bessel functions with argument proportional to $\sqrt{i}$. In terms of these functions the pressure coefficient C is

$$C = 1 + \rho a \ \mathrm{ker}' \ \rho a \qquad (6'')$$

Definitions and tables of the Kelvin functions ber, bei, ker, kei and their derivatives may be found conveniently in reference 4.

At this point it is well to consider the actual behaviour of the pressure coefficient C as a function of $\rho a = \sqrt[4]{k/D}\ a$. From Table 6 of reference 3 or pages 240-241 of reference 4 the following behaviour is obtained.

Table 1. Values of pressure coefficient C

ρa	0	0.3	0.5	0.7	1.0	1.5	2.0	∞
C	0	0.034	0.090	0.166	0.305	0.55	.787	1

It would seem to be the object of the flexible surface layer to reduce appreciably the maximum value of the load pressure on the subsoil which would be existing without the surface layer. Accordingly, for the flexible surface layer to be significant, pressure coefficients C quite appreciably less than unity would seem to be required. This, according to Table 1, reduces the range of physically significant values of $\rho a = \sqrt[4]{k/D}\ a$ to values not too much in excess of unity. Furthermore, returning to equation (5), it is only as long as $\rho r \leq \rho a$ is not more than about unity that the deflection w inside the load area is substantially uniform, because only then do we have that $J_0(\lambda r)$ does not differ much from unity. The foregoing considerations suggest that technical application of flexible surface layers be restricted to values of ρa about equal to unity. Additional reasons for this restriction will become apparent in the section of this report following the present section.

Polynomial approximation. As long as ρa is not larger than about unity it is possible to clarify the meaning of the pressure coefficient C as given by equation (6) by introducing into (6") for ker' ρa the first few terms of its series representation in the vicinity of the point $\rho a = 0$. Formula 824.5 of reference 4 leads to the following approximate expression for C,

$$C \approx \frac{1}{8}\pi(\rho a)^2 \qquad (6''')$$

Equation (6''') should not be used for values of ρa greater than unity. It is however useful for smaller values of ρa. We may write (6''') with

$\rho = \sqrt[4]{k/D} = \sqrt[4]{k\,12(1-\nu^2)/Eh^3}$ in the following instructive explicit form

$$c \approx \frac{\pi}{4}\sqrt{3(1-\nu^2)}\sqrt{\frac{k}{E}}\,\frac{a^2}{h^{3/2}} \qquad (6'''')$$

In equation (6'''') the quantities at our disposal are modulus E and thickness h of the surface layer. It is seen that increasing the thickness h means reducing the pressure coefficients (as could hardly be expected otherwise) and likewise increasing the stiffness of the surface layer material, that is making E larger, reduces the value of the pressure coefficient. Of course, the use of the simple formula (6'''') requires that

$$\rho a = \sqrt[4]{12(1-\nu^2)}\sqrt[4]{\frac{k}{E}}\,\frac{a}{h^{3/4}} \lesssim 1 \qquad (7)$$

but this, as stated before, should roughly cover the useful range of a flexible surface layer.

Bending stresses. Explicit values for bending stresses in accordance with equations (4) will be given here only for the center r = 0 where their values would be expected to be largest. Moreover, when r = 0 the values of radial and circumferential bending stress are the same. Introduction of w from (5) into (4) gives

$$\sigma_B(0) = \frac{Eh}{1-\nu}\,\frac{i\pi\lambda^2}{16}\,\frac{p_0}{k}\left[\mu H_1^{(1)}(\mu) + \bar{\mu}H_1^{(2)}(\bar{\mu})\right] \qquad (8)$$

where for brevity

$$\sigma_B(0) = \sigma_{rB}(0) = \sigma_{\theta B}(0).$$

Equation (8) may also somewhat more conveniently be expressed in terms of Kelvin functions. The result is

$$\sigma_B(0) = \frac{\sqrt{3}}{2}\sqrt{\frac{1+\nu}{1-\nu}}\;p_0\sqrt{\frac{E}{kh}}\;\rho a\,\mathrm{kei}'\rho a \qquad (8')$$

Restricting attention again to the cases for which ρa is less than about unity we can use the first few terms of the series (824.6) of reference 4 for kei' ρa and approximate for ρa kei' ρa in formula (8') as follows,

$$\rho a\,\mathrm{kei}'\rho a \approx \frac{1}{2}(\rho a)^2\log(2/\rho a)$$

Therewith

$$\frac{\sigma_B(0)}{p_0} \approx \frac{3(1+\nu)}{2}\left(\frac{a}{h}\right)^2\log\left[\sqrt[4]{\frac{4}{3(1-\nu^2)}}\;\sqrt[4]{\frac{Eh^3}{ka^4}}\right] \qquad (8'')$$

Use of the simple formula (8'') is allowed as long as the inequality (7) is satisfied, that is, as long as

$$\sqrt[4]{\frac{Eh^3}{ka^4}} \geq \sqrt[4]{12(1-\nu^2)} \approx 2 \qquad (7')$$

Inspection of (8'') reveals that as the thickness h of the surface layer decreases the bending stress increases. Furthermore the bending stress increases as the modulus E of the surface layer increases.

Referring to the earlier result on the foundation pressure stress $\sigma_F(0)$ it is seen that increasing h is beneficial both for σ_F and σ_B while increasing E is beneficial for σ_F but detrimental for σ_B. This result is, as it should be, in agreement with data contained in Figures 1 to 4 of reference 5. The only way in which the present results may at this point be considered to go beyond these earlier results is in the simple explicit analytical formulas (6''''), (7), and (8'').

As a final remark we add that the formulas (8) for $\sigma_B(0)$ are in agreement, for a suitably chosen value of Poisson's ratio ν, with the results in columns 1 and 2 in Table 8 of reference 1. The function F_3 in column 2 is related to the present formulas (8) as follows.

$$F_3 = \frac{1+\nu}{16}\left[\mu H_1^{(1)}(\mu) + \bar{\mu} H_1^{(2)}(\bar{\mu})\right]$$

or

$$F_3 = \frac{1+\nu}{4\pi}\,\frac{a}{l}\,\mathrm{kei}'\,\frac{a}{l}\,.$$

Numerical example. Assumptions:

a = 9 in.	E = 10^5 psi.
h = 4.5 in.	k = 10^2 pci.

Therewith

$$\rho a = \sqrt[4]{\frac{10^2}{\frac{1}{12}10^5 \times (4.5)^3}}\; 9 = \frac{4.82}{5} = .96$$

From Table 1 (equation 6'') for foundation pressure coefficient C,

$$C = \sigma_F(0)/p_0 \approx \underline{0.3}$$

From equation (8') for bending stress $\sigma_B(0)$,

$$\frac{\sigma_B(0)}{p_0} = \sqrt{\frac{3(1+\nu)}{4(1-\nu)}}\;\sqrt{\frac{10^5}{10^2\cdot 4.5}}\;\; .96 \times .35 \approx \underline{5.5}$$

$$(\text{setting } \nu = \tfrac{1}{4})$$

As ρa satisfies the restriction (7) reasonable values of $\sigma_F(0)$ and of $\sigma_B(0)$ should follow from the approximate formulas (6'''') and (8''). From (6'''')

$$C = \frac{\sigma_F(0)}{p_0} \approx \frac{\pi}{4}\sqrt{3}\sqrt{\frac{10^2 \times 4.5}{10^5}}\left(\frac{9}{4.5}\right)^2 \approx \frac{5.38}{15} \approx \underline{\underline{.36}}$$

and (with $\nu = \frac{1}{4}$) from (8'')

$$\frac{\sigma_B(0)}{p_0} \approx \frac{3(1+\nu)}{2}\left(\frac{9}{4.5}\right)^2 \log\left[\sqrt[4]{\frac{10^5}{4.5 \times 10^2}}\,\frac{1.07}{2}\right]$$

$$= \frac{3}{2}\cdot\frac{5}{4}\cdot 4 \quad \log(2.07) \approx \underline{\underline{5.47}}$$

Clearly, the numerical data following from the approximate formulas are quite adequate, even though one is close to the upper end of what was stated to be their usable range.

Infinite Elastic Plate Carrying a Load Distributed Parabolically over a Finite Circular Area

Differential equation (1) again applies with the load distribution p given by

$$p = \begin{cases} p_0\left[\eta + 2(1-\eta)(r/a)^2\right], & r \leqslant a \\ 0, & a < r \end{cases} \tag{9}$$

The parameter η may have any real value whatsoever. The range from zero to two is of particular interest. When $\eta = 1$ the load distribution is uniform. When $\eta = 0$ it rises from no load at midpoint to maximum load at the outer rim of the load surface. When $\eta = 2$ the load intensity is greatest for $r = 0$ and decreases to zero at $r = a$. The total load

$\int_0^a 2\pi r p dr = \pi a^2 p_0$ is independent of the value of η.

The following formulas are obtained after considerable transformations

$$\frac{\sigma_F(0)}{p_0} = 1 + \rho a \,\mathrm{ker}'\rho a$$
$$+ (1-\eta)\left[-1 + \rho a\,\mathrm{ker}'\rho a + \frac{8\mathrm{kei}'\rho a}{\rho a} - 4\mathrm{ker}\,\rho a\right] \tag{10}$$

$$\frac{\sigma_B(0)}{p_0} = \frac{\sqrt{3}}{2}\sqrt{\frac{1+\nu}{1-\nu}}\sqrt{\frac{E}{Kh}}\left\{\rho a\,\mathrm{kei}'\rho a\right.$$
$$\left.(1-\eta)\left[\rho a\,\mathrm{kei}'\rho a - 4\mathrm{kei}\,\rho a - \frac{8\mathrm{ker}'\rho a}{\rho a} - \frac{8}{(\rho a)^2}\right]\right. \tag{11}$$

When $\eta = 1$ then equation (10) reduces to (6'') and equation (11) reduces to (8') as it should. We may write

$$\frac{\sigma_F(0)}{p_0} = C_F + (1-\eta)C_F^* \tag{10'}$$

and

$$\frac{\sigma_B(0)}{p_0} = \left[C_B + (1-\eta)C_B^*\right]\frac{\sqrt{3}}{2}\sqrt{\frac{1+\nu}{1-\nu}}\,p_0\sqrt{\frac{E}{Kh}} \tag{11'}$$

In both formulas the second terms on the right describe the effect of non-uniformity of the load distribution. In order to be sure that the results for uniform load distribution are significant we should, it would appear, require that these second terms be negligible compared with the first terms. The following short table supplies information on this point.

Table 2. Relative contributions to σ_F and σ_B

ρa	0	0.5	1.0	2.0
C_F	0	.09	.30	.79
C_F^*	0	-.002	-.02	-.17
C_B	0	.16	.35	.44
C_B^*			-.07	-.33

Inspection of Table 2 reveals that when ρa is appreciably larger than unity then the effect of non-uniformity of the applied load can no longer be neglected. Since the actual distribution may not be uniform it follows that calculations based on uniformity of load distribution should be viewed with caution as soon as ρa is greater than about unity.

Infinite Elastic Plate Subjected to Uniform Tension in the Plane of the Plate and Carrying a Uniformly Distributed Lateral Load

Let σ be the two-dimensional hydrostatic state of tension in the undeflected surface layer. Let $N = h\sigma$ be the corresponding stress resultants. The effect of this state of tension on the resistance against lateral loads is expressed by modifying the differential equation for the deflection w, as follows:

$$D\nabla^2\nabla^2 w - N\nabla^2 w + kw = p \tag{12}$$

Attention will be restricted to the case of a uniformly distributed lateral load p, as given by equation (2).

The solution w is found to be of the following form

$$w = \begin{cases} \frac{p_0}{k}\left[1 + C_1 J_0(\lambda r) + \bar{C}_1 J_0(\bar{\lambda} r)\right], & r \leq a \\ \\ \frac{p_0}{k}\left[C_2 H_0^{(1)}(\lambda r) + \bar{C}_2 H^{(2)}(\bar{\lambda} r)\right], & a < r \end{cases} \tag{13}$$

where

$$\lambda = \rho e^{i\phi}, \qquad \bar{\lambda} = \rho e^{-i\phi}, \tag{14a}$$

$$\rho = \sqrt[4]{\frac{k}{D}}, \qquad \cos 2\phi = -\frac{N}{2\sqrt{kD}} \tag{14b}$$

and

$$C_1 = \frac{\pi\mu}{4i}\,\frac{H_1^{(1)}(\mu)}{1 + (N/2D\bar{\lambda}^2)} \tag{15a}$$

$$C_2 = \frac{\pi \mu}{4i} \frac{J_1(\mu)}{1 + (N/2D\bar{\lambda}^2)} \tag{15b}$$

$$\mu = \lambda a = \rho a e^{i\phi} \tag{14c}$$

For the numerical evaluation of the effect of the built-in tensile stress $\sigma = N/h$ we restrict attention to the values of the foundation pressure at the origin, $\sigma_F(0)$, and to the bending stress in the plate at the origin, $\sigma_B(0)$. The following expressions are obtained for these two quantities

$$\frac{\sigma_F(0)}{p_0} = 1 + \frac{\pi \rho a}{4}\left[\frac{u_1 - V_1}{\sin\phi} + \frac{v_1 + U_1}{\cos\phi}\right] \tag{16}$$

$$\frac{\sigma_B(0)}{p_0} = \pm\sqrt{\frac{3(1+\nu)}{4(1-\nu)}}\sqrt{\frac{E}{kh}}\,\frac{\pi \rho a}{4}\left[\frac{u_1 - V_1}{\sin\phi} - \frac{v_1 + U_1}{\cos\phi}\right] \tag{17}$$

The quantities u_1, v_1, U_1, V_1 occurring in (16) and (17) are tabulated functions defined in accordance with reference 3 as follows:

$$\left.\begin{aligned} J_1(\mu) &= u_1(\rho a, \phi) + i v_1(\rho a, \phi) \\ Y_1(\mu) &= U_1(\rho a, \phi) + i V_1(\rho a, \phi) \\ H_1^{(1)}(u) &= (u_1 - V_1) + i(v_1 + U_1) \end{aligned}\right\} \tag{18}$$

In the absence of built-in tension, that is for N = 0, we have according to (14b) $\cos 2\phi = 0$, $\phi = \pi/4$ and the results reduce to those previously obtained for this case (pages 7 to 13). The solution (13) and (15) is valid in the form stated as long as $|\cos 2\phi| < 1$, that is as long as $N \leq 2\sqrt{kD}$. We shall find by numerical examples that this relation is satisfied for all practically possible values of N. However, in the event that we wished to deduce from our result the limiting case of a "membrane" with D = 0 we would have to write our results first in a different form from the present one.

The use of the formulas (16) and (17) and the influence of pre-tension on stresses will be illustrated by a numerical example as follows.

Numerical example. Assumptions:

a = 9 in.	$k = 10^2$ pci.
h = 4.5 in.	$E = 10^5$ psi.

Therewith, according to (14),

$$\rho a = \sqrt[4]{\frac{10^2}{\frac{1}{12}10^5 \times 4.5^3}}\; 9 = .96$$

$$\cos 2\phi = \frac{-\sigma\, 4.5}{2\sqrt{10^2 \times \frac{1}{12}10^5 \times 4.5^3}} = \frac{-\sigma}{3860}$$

We restrict attention to cases for which σ is of the order of hundreds psi at most. We have then $|\cos 2\phi| \ll 1$ or $2\phi \approx \pi/2$ and $\phi \approx \pi/4$. We take that value of ϕ just beyond $\pi/4$ for which the functions in the expressions for σ_F and σ_B are tabulated. This means that we take

$$\phi = 5\pi/18 = 50^\circ$$

and

$$\cos 2\phi = \cos 100^\circ = -\cos 80^\circ = -.1736$$

Therewith

$$\sigma = 3860 \times .1736 = 670 \text{ psi.}$$

We need further

$$\sin 50^\circ = .766 \qquad \cos 50^\circ = .643$$

and*

$$u_1(.96, 50^\circ) = +.356$$

$$v_1(.96, 50^\circ) = +.338$$

$$U_1(.96, 50^\circ) = -.785$$

$$V_1(.96, 50^\circ) = +.579$$

Therewith, from (16)

$$\frac{\sigma_F(0)}{p_0} = 1 + \frac{.96\pi}{4}\left[\frac{.356 - .579}{.766} + \frac{.338 - .785}{.643}\right] = \underline{\underline{0.26}}$$

which means that the effect of 670 psi pretension is to reduce the maximum foundation pressure by about 13% compared to the value of this pressure in the absence of the pretension.

Next, from (17), setting $\nu = 1/4$,

$$\frac{\sigma_B(0)}{p_0} = \sqrt{\frac{5}{4}}\sqrt{\frac{10^5}{10^2\ 4.5}}\ \frac{.96\pi}{4}\left[\frac{.356 - .579}{.766} - \frac{.338 - .785}{.643}\right] = \underline{\underline{5.25}}$$

which means that the effect of pretension in this case reduces the maximum bending stress by about 5%.

While further numerical examples can readily be calculated in the same manner as for the foregoing case it is apparent that pretension stresses of some hundreds of psi have only a relatively minor influence on foundation pressure and bending stress in a flexible surface layer in the range of dimensions and rigidities which are under consideration.

Analysis of Non-linear Effects

Analysis of non-linear effects is to be based on the differential equations for finite deflections of elastic plates. These are, for the case of rotationally symmetric displacements,

$$D\nabla^2\nabla^2 w + kw = \frac{1}{r}\frac{d}{dr}\left[\frac{dF}{dr}\frac{dw}{dr}\right] + p \tag{19a}$$

$$\nabla^2\nabla^2 F = -\frac{Dh}{2r}\frac{d}{dr}\left[\left(\frac{dw}{dr}\right)^2\right] \tag{19b}$$

---

*U_1 and V_1 are on page 203 of reference 3. The quantities u_1 and v_1 are on page 203 of reference 6.

Here

$$p = \begin{cases} p_0, & r \leq a \\ 0, & r > a \end{cases} \tag{2}$$

The quantities D, ∇^2, w, k, p and Eh are the same as those used in the linear small-deflection theory. The function F represents the stresses in the mid-plane of the plate which are absent for practical purposes for sufficiently small deformations. The above equations correspond to those considered on pages 52-70 of reference 1 except that here horizontal subgrade reaction is considered negligible.

A natural procedure of solving (19a) and (19b) consists in the following iterative procedure.

(1) Solve the first of the two equations for w as if the non-linear term involving F were absent.

(2) With the value of w so obtained go into the second equation and obtain F in terms of w.

(3) With these values of F and w compute the non-linear term in the first differential equation. If this term, computed in this manner, is small compared with p then the non-linear effect is negligible.

A systematic approach by means of the introduction of suitable dimensionless variables can furnish important qualitative information in this problem without performance of any numerical calculations. Let a be radius of the load surface and assume that a is a "representative" length in the sense that significant changes are assumed to take place over distances of the order a. Observe further that if a is a representative distance then in the linear theory the bending term $D\Delta^2\Delta^2 w$ is of importance and the deflection is of order of magnitude pa^4/D, just as if it were a plate of radius a supported along the edge, instead continuously by the subgrade. On the basis of these observations write

$$r = a\xi \tag{20a}$$

$$w = \frac{p_0 a^4}{D} f(\xi) \tag{20b}$$

where now f is "of order unity". Further

$$\nabla^2 w = \frac{p_0 a^2}{D} \nabla_*^2 f(\xi) \tag{20c}$$

where

$$\nabla_*^2 = \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} \tag{20d}$$

Because of the disposition concerning a we have that $\nabla_*^2 f$ is also "of order unity."

Since we want to obtain F in first approximation from the second differential equation we must non-dimensionalize F in such manner that the terms with F and w in this equation are of the same importance. Inspection reveals that this can be accomplished by setting

$$F = Eh\left(\frac{p_0 a^4}{D}\right)^2 g(\xi) \tag{20c}$$

The system of differential equations to be considered is now of the form

$$\nabla_*^2\nabla_*^2 f + \frac{ka^4}{D} f = \frac{Eh}{D}\left(\frac{p_0 a^4}{D}\right)^2 \frac{1}{\xi}\frac{d}{d\xi}\left[\frac{dg}{d\xi}\frac{df}{d\xi}\right] + \frac{p}{p_0} \tag{21a}$$

$$\nabla_*^2\nabla_*^2 g = -\frac{1}{2}\frac{1}{\xi}\frac{d}{d\xi}\left[\left(\frac{df}{d\xi}\right)^2\right] \tag{21b}$$

where

$$\frac{p}{p_0} = \begin{cases} 1, & \xi < 1 \\ 0, & 1 < \xi \end{cases} \tag{21c}$$

There are still two parameters in these equations, ka^4/D and $(Eh/D)(p_0a^4/D)^2$. We know from the earlier work concerning small deflections that ka^4/D should be no more than unity. (We can now say that it should be no larger than this in order that the flexural terms in the differential equation can make their contribution.) The remaining parameter $(Eh/D)(p_0a^4/D)^2$ indicates whether or not non-linearity is important. If $(Eh/D)(p_0a^4/D)^2$ is of "order unity" then non-linearity is important. However, if

$$\frac{Eh}{D}\left(\frac{p_0 a^4}{D}\right)^2 \ll 1$$

then non-linearity is not important. This condition may be expressed in terms of deflections. Set

$$\frac{p_0 a^4}{D} = \delta$$

when ∇ is the order of the deflection according to the linear theory. Further $Eh/D = 12(1 - \nu^2)/h^2$. Accordingly the above condition may be written in the form

$$12(1 - \nu^2)\frac{\delta^2}{h^2} \ll 1.$$

In other words, non-linear effects are negligible, as long as the deflection ∇ is small compared with the thickness h of the plate. This result however requires for its validity that ka^4/D be not large compared with unity. If ka^4/D is large compared with unity, a case not considered important here, then a different procedure must be used for the analysis of the problem.

It may finally be stated that in the event that non-linear corrections to the linear theory might be desired an appropriate systematic procedure would be to develop f and g in powers of the parameter $(Eh/D)(p_0a^4/D)^2$.

CONCLUSIONS

Comparative studies of flexible surface layers endowed with a variety of properties may be carried out most simply and with errors of tolerable magnitude by representing the subgrade by a system of tension-compression springs. This type of subgrade has the same behaviour as a "heavy liquid" subgrade characterized by a foundation constant k.

As long as the ratio of diameter of load surface to thickness of surface layer, 2a/h, is about unity or larger it is permissible to consider the surface layer as a "thin plate." For values of 2a/h between about one and two,

transverse shear deformation in the plate is of quantitative significance. It is recommended that at some future date the work of the present report be supplemented so as to coordinate the present results with results showing the correction effect of transverse shear deformation.

The present report suggests that the practical usefulness of a flexible surface layer requires satisfaction of the limit relation

$$\frac{2a}{h} \sqrt[4]{\frac{kh}{E}} < 1.5 \tag{I}$$

where E is the modulus of elasticity of the surface layer material and the other quantities are defined earlier.

As long as the above limit relation is satisfied one has as simple formulas representing maximum foundation pressure $\sigma_F(0)$ and maximum tension (bending) stress in the surface layer $\sigma_B(0)$ with no more than a few percent error,

$$\frac{\sigma_F(0)}{p_0} = \frac{\pi}{4} \sqrt{3(1 - \nu^2)} \left(\frac{a}{h}\right)^2 \sqrt{\frac{kh}{E}} \tag{II}$$

and

$$\frac{\sigma_B(0)}{p_0} = \frac{3(1 + \nu)}{8} \left(\frac{a}{h}\right)^2 \log \left[\frac{4}{3(1 - \nu^2)} \frac{E}{kh} \left(\frac{h}{a}\right)^4\right] \tag{III}$$

In the foregoing two formulas the quantity ν is Poisson's ratio of the surface layer material and p_0 is the average loading stress, the total load on the surface layer being $\pi a^2 p_0$. (It is noted that the maximum deflection w(0) is related to $\sigma_F(0)$ in the form $w = \sigma_F/k$.)

As long as the limit relation (I) is satisfied we have the following assurances, as is shown in the body of the present report,

(1) The loading stress on the subgrade is less than half of what it would be if the load were applied to the subgrade without the interposition of the flexible surface layer.

(2) Non-uniformity of the loading stress distribution is of secondary importance.

(3) Built-in tensions of reasonable magnitude in the flexible surface layer are of negligible effect.

(4) Non-linear effects due to finite deformation are negligible as long as the transverse deflection due to load of the surface layer is smaller than about one half the thickness of the surface layer.

ACKNOWLEDGMENT

The present paper is based on a report prepared for the Engineer Research and Development Laboratories at Fort Belvoir, Virginia, under Contract DA-44-009 ENG-1863, the final report having been accepted March, 1954.

The author wishes to express his appreciation for helpful discussions on the subject of this paper to Mr. J.H. Reynolds, Jr., at that time Project Engineer at Fort Belvoir, and to his colleagues Drs. T. W. Lambe, G.E. Murray, and D.W. Taylor of M.I.T.'s Soil Mechanics group.

REFERENCES

1. "Deflections, moments and reactive pressures for concrete pavements," by G. Pickett, M.E. Raville, W.C. Jones and F.J. McCormick, Kansas State College Bulletin No. 65, October, 1951.

2. "On the bending of axially symmetric plates on elastic foundations," by P.M. Naghdi and J.C. Rowley, Proc. 1st Midwestern Conf. Solid Mech. 1953, pp. 119-123.

3. "Table of the Bessel functions $Y_0(z)$ and $Y_1(z)$ for complex arguments," by the Computation Laboratory of the National Bureau of Standards, Columbia University Press, 1950.

4. "Tables of integrals and other mathematical data." by H.B. Dwight, The McMillan Company, 1947.

5. "Design criteria for expediently solidified flexible soil surfaces." Memorandum by J.H. Reynolds, Jr., E.R.D.L., Fort Belvoir, Virginia, October, 1953.

6. "Table of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments," by the Mathematical Tables Project of the National Bureau of Standards, Columbia University Press, 1947.

7. "Analytical studies of landing mats for airfields." by G. Pickett, Report of Eng. Exp. Station, Kansas State College, Manhattan, Kansas, December 1951.

APPENDIX

A. Infinite elastic plate carrying load distributed parabolically. Derivations.
The following derivations contain the case of uniformly distributed load as a special case.

Differential equation

$$D\nabla^2\nabla^2 w + kw = \begin{cases} p_0\left[\eta + 2(1-\eta)r^2/a^2\right], & r \leq a \\ 0, & a < r \end{cases} \quad (1)$$

Total load

$$2\pi\int_0^{\infty} p\, r\, dr = 2\pi p_0 \int_0^a \left[\eta + 2(1-\eta)\frac{r^2}{a^2}\right] r\, dr$$

$$= 2\pi p_0\left[\eta\frac{a^2}{2} + 2(1-\eta)\frac{a^2}{4}\right] = \pi p_0 a^2, \quad (2)$$

is independent of η

Transition conditions

$r = a$: w, w', M_r, Q_r continuous

where

$$M_r = -D(w'' + \nu r^{-1} w'), \quad Q_r = -D(\nabla^2 w)'$$

Can be written in the form

$$r = a: \quad w,\ w',\ \nabla^2 w,\ (\nabla^2 w)' \ \text{continuous.} \tag{3}$$

General solution of differential equation

$$w = w_{hom} + w_{part} \tag{4}$$

$$w_{hom} = A_1 J_0(\lambda r) + A_2 Y_0(\lambda r) + A_3 J_0(\bar{\lambda} r) + A_4 Y_0(\bar{\lambda} r) \tag{5}$$

$$w_{part} = \begin{cases} (p_0/k)\left[\eta + 2(1-\eta)(r^2/a^2)\right], & r \leq a \\ 0, & a < r \end{cases} \tag{6}$$

where

$$\lambda = \sqrt{i}\ \sqrt[4]{\frac{k}{D}} = \frac{1+i}{\sqrt{2}}\ \sqrt[4]{\frac{k}{D}}$$

$$\bar{\lambda} = \frac{1-i}{2}\ \sqrt[4]{\frac{k}{D}}, \quad \sqrt[4]{\frac{k}{D}} = \rho \tag{7}$$

Selection of suitable constants of integration

Since $Y_0(0) = \infty$ we must set A_2 and A_4 equal to zero in the region $r \leq a$. Furthermore, according to references 3 and 6, $J_0(\lambda r)$ and $J_0(\bar{\lambda} r)$ are conjugate complex. In accordance with reference 3

$$J_0(\lambda r) = u_0(\rho r, \pi/4) + i v_0(\rho r, \pi/4)$$
$$J_0(\bar{\lambda} r) = u_0(\rho r, \pi/4) - i v_0(\rho r, \pi/4) \tag{8}$$

where u_0 and v_0 are tabulated as functions of ρr.

In order to have w a real function we write, for $r \leq a$,

$$w = \frac{p_0}{k}\left[C_1 J_0(\lambda r) + \bar{C}_1 J_0(\bar{\lambda} r) + \eta + 2(1-\eta)\frac{r^2}{a^2}\right] \tag{9}$$

where C_1 and $\bar{C}_1$ are conjugate complex.

In order to obtain the proper form for w in the region $a < r$ it is necessary to consider the behaviour of the Bessel functions for large values of their arguments. We write, in accordance with usage,

$$J_0 + i Y_0 = H_0^{(1)}, \qquad J_0 - i Y_0 = H_0^{(2)} \tag{10}$$

and observe that since

$$Y_0(\lambda r) = U_0 + i V_0 \qquad Y_0(\bar{\lambda} r) = U_0 - i V_0 \tag{11}$$

we have that

$$H_0^{(2)}(\bar{\lambda} r) = \overline{H_0^{(1)}(\lambda r)} \tag{12}$$

For large values of the argument

$$H_0^{(1)}(z) \sim \frac{1}{\sqrt{z}} e^{iz}, \quad H_0^{(2)}(z) \sim \frac{1}{\sqrt{z}} e^{-iz} \tag{13}$$

and consequently

$$H_0^{(1)}(\lambda r) \approx \frac{1}{\sqrt{\lambda r}}\, e^{i(1+i)\frac{\rho r}{\sqrt{2}}} = \frac{1}{\sqrt{\lambda r}}\, e^{(-1+i)\frac{\rho r}{\sqrt{2}}}$$

$$H_0^{(2)}(\bar{\lambda} r) \approx \frac{1}{\sqrt{\bar{\lambda} r}}\, e^{-i(1-i)\frac{\rho r}{\sqrt{2}}} = \frac{1}{\sqrt{\bar{\lambda} r}}\, e^{(-1-i)\frac{\rho r}{\sqrt{2}}} \tag{14}$$

Both these functions decay exponentially for large ρr, while $H_0^{(1)}(\lambda r)$ and $H_0^{(2)}(\lambda r)$ increase exponentially. The latter functions must therefore be excluded from the solution of the present problem. Altogether we have then a as a real solution valid in $a < r$,

$$w = \frac{p_0}{k}\left[C_2 H_0^{(1)}(\lambda r) + \bar{C}_2 H_0^{(2)}(\bar{\lambda} r)\right] \tag{15}$$

where C_2 and $\bar{C}_2$ are conjugate complex.

Bessel function formulas

For evaluation of transition conditions it is necessary to utilize the following differentiation formulas

$$(J_0, Y_0)'(\lambda r) = -\ (J_1, Y_1)(\lambda r) \tag{16}$$

$$\nabla^2 (J_0, Y_0)(\lambda r) = -\ \lambda^2 (J_0, Y_0)(\lambda r) \tag{17}$$

Evaluation of transition conditions (3)

Set

$$\lambda a = \mu, \quad \bar{\lambda} a = \bar{\mu}$$

With w from (9) and (15)

$$\begin{aligned}
&C_1 J_0(\mu) + \bar{C}_1 J_0(\bar{\mu}) + (2 - \eta) = C_2 H_0^{(1)}(\mu) + \bar{C}_2 H_0^{(2)}(\bar{\mu}) \\
&C_1 \mu J_1(\mu) + \bar{C}_1 \bar{\mu} J_1(\bar{\mu}) - 4(1 - \eta) = C_2 \mu H_1^{(1)}(\mu) + C_2 \bar{\mu} H_1^{(2)}(\bar{\mu}) \\
&C_1 \mu^2 J_0(\mu) + \bar{C}_1 \bar{\mu}^2 J_0(\bar{\mu}) - 8(1 - \eta) = C_2 \mu^2 H_0^{(1)}(\mu) + \bar{C}_2 \bar{\mu}^2 H_0^{(2)}(\bar{\mu}) \\
&C_1 \mu^3 J_1(\mu) + \bar{C}_1 \bar{\mu}^3 J_1(\bar{\mu}) = C_2 \mu^3 H_1^{(1)}(\mu) + \bar{C}_2 \bar{\mu}^3 H_1^{(2)}(\bar{\mu})
\end{aligned} \tag{19}$$

In view of (7) and (18) we have that $\bar{\mu}^2 = -\mu^2$. This fact permits considerable simplification of (19). Combine the first and third of these equations as follows by suitable addition

$$2C_1 J_0(\mu) + (2 - \eta) - \frac{8(1 - \eta)}{\mu^2} = 2C_2 H_0^{(1)}(\mu) \tag{20a}$$

Combine second and fourth equations as follows by suitable addition

$$2C_1 \mu J_1(\mu) - 4(1 - \eta) = 2C_2 \mu H_1^{(1)}(\mu) \tag{20b}$$

or

$$\left.\begin{aligned} C_1 J_0(\mu) - C_2 H_0^{(1)}(\mu) &= -(1 - \tfrac{\eta}{2}) + \frac{4}{\mu^2}(1 - \eta) \\ C_1 J_1(\mu) - C_2 H_1^{(1)}(\mu) &= \frac{2}{\mu}(1 - \eta) \end{aligned}\right\} \quad (20')$$

From this

$$\Delta C_1 = \left[(1 - \tfrac{\eta}{2}) - \frac{4}{\mu^2}(1 - \eta) \right] H_1^{(1)}(\mu) + \frac{2}{\mu}(1 - \eta) H_0^{(1)}(\mu) \quad (21a)$$

$$\Delta C_2 = \frac{2}{\mu}(1 - \eta) J_0(\mu) + \left[(1 - \tfrac{\eta}{2}) - \frac{4}{\mu^2}(1 - \eta) \right] J_1(\mu) \quad (21b)$$

where

$$\Delta = J_1(\mu) H_0^{(1)}(\mu) - J_0(\mu) H_1^{(1)}(\mu) \quad (21c)$$

The determinant Δ may be simplified by means of the identity

$$J_0(z) Y_1(z) - J_1(z) Y_0(z) = -2/\pi z$$

(see reference 3, page XXIII). Write (21c) with (10),

$$\Delta = J_1 \left[J_0 + i Y_0 \right] - J_0 \left[J_1 + i Y_1 \right] = i \left[J_1 Y_0 - J_0 Y_1 \right]$$

or

$$\Delta = 2i/\pi\mu \quad (21d)$$

Therewith,

$$C_1 = \frac{\pi}{2i} \left\{ \left[(1 - \tfrac{\eta}{2})\mu - \frac{4(1-\eta)}{\mu} \right] H_1^{(1)}(\mu) + 2(1 - \eta) H_0^{(1)}(\mu) \right\} \quad (22a)$$

$$C_2 = \frac{\pi}{2i} \left\{ \left[(1 - \tfrac{\eta}{2})\mu - \frac{4(1-\eta)}{\mu} \right] J_1(\mu) + 2(1 - \eta) J_0(\mu) \right\} \quad (22b)$$

For uniform load distribution for which $\eta = 1$ these expressions simplify to

$$\eta = 1, \quad \begin{cases} C_1 = \frac{1}{4} i \pi \mu H_1^{(1)}(\mu) & (23a) \\ C_2 = \frac{1}{4} i \pi \mu J(\mu) & (23b) \end{cases}$$

Introduction of (23) into (9) and (15) leads to equations (5) in the main body of the report.

Expression for center deflections

Introduce (22) into (9) according to which

$$w(0) = (p_0/k) \left[\eta + C_1 + \bar{C}_1 \right] \quad (24)$$

Expression for foundation pressure $\sigma_F(0)$

$$\frac{\sigma_F(0)}{p_0} = \eta + 2\,\mathrm{Re}\left\{ C_1 \right\} \quad (25)$$

where, with $\sqrt[4]{k/D}\, a = \kappa$,

$$C_1 = \frac{\pi}{2i}\left\{\left[(1 - \frac{\eta}{2})\frac{1+i}{\sqrt{2}}\kappa - \frac{4(1-\eta)}{\kappa}\frac{\sqrt{2}}{1+i}\right] H_1^{(1)}(\kappa e^{i\pi/4})\right.$$

$$\left. 2(1-\eta)\, H_0^{(1)}(\kappa e^{i\pi/4})\right\} \qquad (22a')$$

Now, from (8), (10) and (11),

$$H_0^{(1)}(\kappa e^{i\pi/4}) = J_0(\kappa e^{i\pi/4}) + iY_0(\kappa e^{i\pi/4})$$

$$= u_0(\kappa,\tfrac{\pi}{4}) + iv_0(\kappa,\tfrac{\pi}{4}) + i\left[U_0(\kappa,\tfrac{\pi}{4}) + iV_0(\kappa,\tfrac{\pi}{4})\right]$$

$$= \left[u_0(\kappa,\tfrac{\pi}{4}) - V_0(\kappa,\tfrac{\pi}{4})\right] + i\left[v_0(\kappa,\tfrac{\pi}{4}) + U_0(\kappa,\tfrac{\pi}{4})\right]$$

and in view of page XXIV of reference 3

$$H_0^{(1)}(\kappa e^{i\pi/4}) = -\frac{2}{\pi}\,\mathrm{kei}\,\kappa - \frac{2i}{\pi}\,\mathrm{ker}\,\kappa \qquad (26)$$

Analogously,

$$H_0^{(1)}(\kappa e^{i\pi/4}) = u_1(\kappa,\tfrac{\pi}{4}) - V_1(\kappa,\tfrac{\pi}{4})$$

$$+ i\left[v_1(\kappa,\tfrac{\pi}{4}) + U_1(\kappa,\tfrac{\pi}{4})\right]$$

$$= \frac{2}{\pi}\left[\mathrm{kei}_1\kappa + i\,\mathrm{ker}_1\kappa\right] \qquad (27)$$

and therewith

$$C_1 = \left[(1-\frac{\eta}{2})\frac{1+i}{\sqrt{2}}\kappa - \frac{4(1-\eta)}{\kappa}\frac{1-i}{\sqrt{2}}\right]\left[\mathrm{ker}_1\kappa - i\,\mathrm{kei}_1\kappa\right]$$

$$-2(1-\eta)(\mathrm{ker}\,\kappa - i\,\mathrm{kei}\,\kappa)$$

and

$$\mathrm{Re}\left\{C_1\right\} = (1-\frac{\eta}{2})\frac{\kappa}{\sqrt{2}}(\mathrm{ker}_1\kappa + \mathrm{kei}_1\kappa)$$

$$+\frac{4(1-\eta)}{\sqrt{2}\,\kappa}(-\mathrm{ker}_1\kappa + \mathrm{kei}_1\kappa) - 2(1-\eta)\,\mathrm{ker}\,\kappa$$

Write finally (reference 4, page 190-191)

$\mathrm{ker}_1\kappa = (1/\sqrt{2})(\mathrm{ker}'\kappa - \mathrm{kei}'\kappa)$ and $\mathrm{kei}_1\kappa = (1/\sqrt{2})(\mathrm{ker}'\kappa + \mathrm{kei}'\kappa)$.

Therewith $\quad \mathrm{ker}_1\kappa + \mathrm{kei}_1\kappa = \sqrt{2}\,\mathrm{ker}'k$

$$\mathrm{ker}_1 k - \mathrm{kei}_1\kappa = -\sqrt{2}\,\mathrm{kei}'k$$

and

$$\mathrm{Re}\left\{C_1\right\} = (1 - \frac{\eta}{2})\kappa\,\mathrm{ker}'\kappa + \frac{4(1-\eta)}{\kappa}\mathrm{kei}'\kappa - 2(1-\eta)\mathrm{ker}\,\kappa$$

$$= \frac{1}{2}\kappa\mathrm{ker}'\kappa + (1-\eta)\left[-2\mathrm{ker}\,\kappa + \frac{4}{\kappa}\mathrm{kei}'\kappa + \frac{\kappa}{2}\mathrm{ker}'\kappa\right] \quad (28)$$

Introducing (28) in (25) finally results in the following expression for foundation pressure $\sigma_F(0)$, writing $\kappa = \rho a$,

$$\frac{\sigma_F(0)}{p_0} = 1 + \rho a\,\mathrm{ker}'\rho a + (1-\eta)\left[-4\mathrm{ker}\,\rho a - 1 + \frac{8}{\rho a}\mathrm{kei}'\rho a + \rho a\,\mathrm{ker}'\rho a\right] \quad (29)$$

Equation (29) is the same as equation (10) in the main body of the report and it is discussed there. When $\eta = 1$ then (29) reduces to (6″) in the main body of the report.

Bending stresses

In accordance with equations (4) of the main body of the text

$$\sigma_{rB} = \pm \frac{Eh}{2(1-\nu^2)}\left[\nabla^2 w - \frac{1-\nu}{r}w'\right] \quad (30a)$$

$$\sigma_{\theta B} = \pm \frac{Eh}{2(1-\nu^2)}\left[\frac{1-\nu}{r}w' + \nu\nabla^2 w\right] \quad (30b)$$

Equations (30) will be evaluated for $r = 0$. From equation (9) and in view of (16) and (17), as long as $r \leq a$,

$$w' = -\frac{p_0}{k}\left[C_1\lambda J_1(\lambda r) + \bar{C}_1\bar{\lambda}J_1(\bar{\lambda}r) - \frac{4(1-\eta)r}{a^2}\right] \quad (31a)$$

$$\nabla^2 w = -\frac{p_0}{k}\left[C_1\lambda^2 J_0(\lambda r) + \bar{C}_1\bar{\lambda}^2 J_0(\bar{\lambda}r) - \frac{8(1-\eta)}{a^2}\right] \quad (31b)$$

Since

$$\lim_{r\to 0}\frac{J_1(r)}{r} = \frac{1}{2}, \qquad \lim_{r\to 0} J_0(r) = 1 \quad (32)$$

there follows from (31)

$$\left[\frac{w'(r)}{r}\right]_0 = -\frac{p_0}{2k}\left[C_1\lambda^2 + \bar{C}_1\bar{\lambda}^2 - \frac{8(1-\eta)}{a^2}\right] \quad (33a)$$

$$\left[\nabla^2 w\right]_0 = -\frac{p_0}{k}\left[C_1\lambda^2 + \bar{C}_1\bar{\lambda}^2 - \frac{8(1-\eta)}{a^2}\right] \quad (33b)$$

With $\lambda a = \mu$, $\bar{\lambda}a = \bar{\mu}$ we have from (33) and (30) the following expressions

$$\sigma_{rB}(0) = \mp \frac{Eh}{1-\nu^2}\frac{p_0}{2a^2k}\left[\frac{1+\nu}{2}(\mu^2C_1+\bar{\mu}^2\bar{C}_1) - 4(1+\nu)(1-\eta)\right] \tag{34a}$$

$$\sigma_{\theta B}(0) = \mp \frac{Eh}{1-\nu^2}\frac{p_0}{2a^2k}\left[\frac{1+\nu}{2}(\mu^2C_1+\bar{\mu}^2\bar{C}_1) - 4(1+\nu)(1-\eta)\right] \tag{34b}$$

Equations (34) may be written in the following form

$$\frac{\sigma_B(0)}{p_0} = \frac{\sigma_{rB}(0)}{p_0} = \frac{\sigma_{\theta B}(0)}{p_0}$$

$$= \pm\frac{1+\nu}{4}\frac{Eh}{(1-\nu^2)ka^2}\left[\mu^2C_1+\bar{\mu}^2\bar{C}_1 - 8(1-\eta)\right] \tag{35}$$

Evaluation

With $\mu^2 = i\kappa^2$, from (22a)

$$\mathrm{Re}\left\{\mu^2C_1\right\} = \mathrm{Re}\left\{\frac{\pi\kappa^2}{2}\left[(1-\frac{\eta}{2})\frac{1+i}{\sqrt{2}}\kappa - \frac{4(1-\eta)}{\kappa}\frac{1-i}{\sqrt{2}}\right]H_1^{(1)}(\mu)\right.$$

$$\left. + \kappa^2\pi(1-\eta)H_0^{(1)}(\mu)\right\}$$

$$= \mathrm{Re}\left\{\frac{\pi\kappa^2}{2}\left[(1-\frac{\eta}{2})\frac{1+i}{\sqrt{2}}\kappa - \frac{4(1-\eta)}{\kappa}\frac{1-i}{\sqrt{2}}\right]\right.$$

$$\cdot\frac{2}{\pi}\left[\mathrm{kei}_1\kappa + i\,\mathrm{ker}_1\kappa\right]$$

$$\left. + \pi\kappa^2(1-\eta)(-\frac{2}{\pi})(\mathrm{kei}\,\kappa + i\,\mathrm{ker}\,\kappa)\right\}$$

$$= \kappa^2\left[(1-\frac{\eta}{2})\frac{\kappa}{\sqrt{2}}(\mathrm{kei}_1\kappa - \mathrm{ker}_1\kappa)\right.$$

$$\left. - \frac{4(1-\eta)}{\sqrt{2}\,\kappa}(\mathrm{kei}_1\kappa + \mathrm{ker}_1\kappa) - 2(1-\eta)\mathrm{kei}\,\kappa\right]$$

$$= \kappa^2\left[(1-\frac{\eta}{2})\kappa\,\mathrm{kei}'\kappa - \frac{4(1-\eta)}{\kappa}\mathrm{ker}'\kappa - 2(1-\eta)\mathrm{kei}\,\kappa\right] \tag{36}$$

Introduction of (36) into (35) gives

$$\frac{\sigma_B(0)}{p_0} = \mp\frac{Eh}{4(1-\nu)}\frac{1}{ka^2}\left\{\kappa^3\mathrm{kei}'\kappa\right.$$

$$\left. + (1-\eta)\left[\kappa^3\mathrm{kei}'\kappa - 4\kappa^2\mathrm{kei}\,\kappa - 8\kappa\,\mathrm{ker}'\kappa - 8\right]\right\} \tag{37}$$

Equation (37) is modified to writing

$$\frac{Eh}{4(1-\nu)}\frac{\kappa^2}{ka^2} = \frac{Eh}{4(1-\nu)}\sqrt{\frac{k}{D}}\,\frac{a^2}{ka^2} = \frac{Eh}{4(1-\nu)}\frac{1}{\sqrt{kD}}$$

$$= \frac{Eh}{4(1-\nu)}\sqrt{\frac{12(1-\nu^2)}{Eh^3k}} = \frac{\sqrt{12(1-\nu^2)}}{4(1-\nu)}\sqrt{\frac{E}{hk}} = \sqrt{\frac{3}{4}\,\frac{1+\nu}{1-\nu}\,\frac{E}{k}}$$

Therewith

$$\frac{\sigma_B(0)}{p_0} = \mp\sqrt{\frac{3(1+\nu)}{4(1-\nu)}}\sqrt{\frac{E}{kh}}\left\{\begin{array}{l}\rho a \operatorname{kei}'\rho a \\ +(1-\eta)\Big[\rho a \operatorname{kei}'\rho a - 4\operatorname{kei}\rho a \\ \qquad - \dfrac{8\operatorname{ker}'\rho a}{\rho a} - \dfrac{8}{\rho^2 a^2}\Big]\end{array}\right\} \quad (38)$$

When $1 - \eta = 0$ then (38) reduces to equation (8') of the main text. Otherwise it becomes equation (11) of the main text.

Series developments

From reference 4, pages 187-188 and 184:

$$\operatorname{kei} x = \left(\log\frac{2}{x} - \gamma\right)\left(\frac{1}{4}x^2 - \frac{x^6}{2^6\cdot 36} + \cdots\right) - \frac{\pi}{4}\left(1 - \frac{x^4}{64} + \cdots\right)$$
$$\frac{x^2}{4} - \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^6}{2^6\cdot 36} + \cdots$$

$$\operatorname{ker} x = \left(\log\frac{2}{x} - \gamma\right)\left(1 - \frac{x^4}{64} + \cdots\right) + \frac{\pi}{4}\left(\frac{x^2}{4} - \frac{x^6}{2^6\cdot 36} + \cdots\right)$$
$$- \left(1 + \frac{1}{2}\right)\frac{x^4}{2^4\cdot 4} + \cdots$$

$$\operatorname{kei}' x = \left(\log\frac{2}{x} - \gamma\right)\left(\frac{x}{2} - \frac{x^5}{2^6\cdot 6} + \cdots\right) - \frac{1}{x}\left(\frac{x^2}{4} - \frac{x^6}{2^6\cdot 36} + \cdots\right)$$
$$- \frac{\pi}{4}\left(-\frac{x^3}{16} + \frac{x^7}{2^7\cdot 6\cdot 24} - \cdots\right) + \frac{x}{2} - \frac{11}{6}\,\frac{x^5}{2^5\cdot 2\cdot 6} + \cdots$$

$$\operatorname{ker}' x = \left(\log\frac{2}{x} - \gamma\right)\left(-\frac{x^3}{16} + \cdots\right) - \frac{1}{x}\left(1 - \frac{x^4}{64} + \cdots\right)$$
$$+ \frac{\pi}{4}\left(\frac{x}{2} - \frac{x^5}{2^6\cdot 6} + \cdots\right) - \frac{3}{2}\,\frac{x^3}{2^4} + \cdots$$

Therewith

$$1 + x\operatorname{ker}' x = \frac{\pi}{8}x^2 + \left[-\frac{1}{16}\left(\log\frac{2}{x} - \gamma\right) + \frac{1}{64} - \frac{3}{32}\right]x^4 + \cdots \quad (39)$$

$$2x^{-1}\mathrm{kei}'\,x - \mathrm{ker}\,x = \frac{1}{2} + \left[\frac{\pi}{32} - \frac{\pi}{16}\right]x^2$$

$$+ \left[\left(-\frac{1}{2^5\cdot 6} + \frac{1}{64}\right)\left(\log\frac{2}{x} - \gamma\right)\right.$$

$$\left.\left(\frac{1}{2^5\cdot 36} - \frac{11}{6}\,\frac{1}{2^5\cdot 6} + \frac{3}{2}\,\frac{1}{2^4\cdot 4}\right)\right]x^4 \tag{40}$$

$$\frac{8}{x}\mathrm{kei}'\,x - 4\mathrm{ker}\,x + (x\,\mathrm{ker}'x + 1) - 2$$

$$= 2 - 2 + \left[4\left(\frac{\pi}{32} - \frac{\pi}{16}\right) + \frac{\pi}{8}\right]x^2$$

$$+ \left[4\left(\frac{1}{64} - \frac{1}{2^5\cdot 6}\right) - \frac{1}{16}\right]\left(\log\frac{2}{x} - \gamma\right)x^4$$

$$+ \left[\frac{1}{2^5\cdot 9} - \frac{11}{2^5\cdot 9} + \frac{3}{2^5} + \frac{1}{64} - \frac{3}{32}\right]x^4 + \cdots$$

$$= -\frac{x^4}{48}\left(\log\frac{2}{x} - \gamma\right) - \frac{11x^4}{9\cdot 64} + \cdots\cdot \tag{41}$$

Introduction of (39) to (41) into (29) gives the following expansion for the foundation pressure

$$\frac{\sigma_F(0)}{p_0} = \frac{\pi}{8}(\rho a)^2 + (1 - \eta)\left[-\frac{(\rho a)^4}{48}\left(\log\frac{2}{\rho a} - \gamma + \frac{11}{12}\right) + \cdots\right] \tag{42}$$

When $(1 - \eta) = 0$ then (42) reduces to equation (6''') of the main text. Otherwise it shows that even for $1 - \eta \neq 0$ the foundation pressure behaves as if $1 - \eta = 0$, as long as Δ a is small enough. This is in accordance with the appropriate statement made in the main body of the report.

We finally note that equations (8') and (11) for $\sigma_B(0)/p_0$ may be developed in an entirely analogous manner but omit the details.

B. Infinite elastic plate subjected to uniform tension and carrying a uniform lateral load over a circular area. Derivations.

The following derivations, besides giving important practical information, illustrate the usefulness of the numerical tables in references 3 and 6.

Differential equation

$$D\nabla^2\nabla^2 w + N\nabla^2 w + kw = \begin{cases} p_0, & r \leq a \\ 0, & a < r \end{cases} \tag{1}$$

Solution

$$w = w_h + \begin{cases} p_0/k, & r \leq a \\ 0, & a < r \end{cases} \tag{2}$$

Where w_h is the general solution of the homogeneous equation (1).

Determination of w_h

For brevity the subscript "h" will be omitted. The homogeneous equation (1) may be written in the form

$$(\nabla^2 + \lambda^2)(\nabla^2 + \bar{\lambda}^2)w = 0 \tag{3}$$

where

$$\lambda^2 \bar{\lambda}^2 = k/D, \qquad \lambda^2 + \bar{\lambda}^2 = - N/D. \tag{4}$$

Let

$$\lambda^2 = \rho^2 e^{2i\phi}, \qquad \bar{\lambda}^2 = \rho^2 e^{-2i\phi} \tag{5}$$

Then, as in the case of absent initial tension,

$$\rho = \sqrt[4]{k/D} \tag{6}$$

but now

$$2\rho^2 \cos 2\phi = - N/D \tag{7}$$

or

$$\cos 2\phi = - \frac{N}{2\sqrt{kD}} \tag{8}$$

The general solution of (3) is composed of the general solutions of the two equations $(\nabla^2 + \lambda^2)f = 0$ and $(\nabla^2 + \bar{\lambda}^2)f = 0$. Accordingly, if at the same time we make sure that all solutions are in real form,

$$w = C_1 J_0(\lambda r) + \bar{C}_1 J_0(\bar{\lambda} r)$$

$$+ C_2 Y_0(\lambda r) + \bar{C}_2 Y_0(\bar{\lambda} r) \tag{9}$$

Tabulated form of Bessel functions in (9)

According to references 3 and 6,

$$J_0(\rho r e^{\pm i\phi}) = u_0(\rho r, \phi) \pm i v_0(\rho r, \phi) \tag{10a}$$

$$Y_0(\rho r e^{i\phi}) = U_0(\rho r, \phi) \pm i V_0(\rho r, \phi) \tag{10b}$$

Approximate value of angle ϕ

According to (8) when N - 0 then $\phi = \pi/4$. When $N > 0$ then $\cos 2\phi < 0$ and consequently $2\phi > \pi/2$ or $\phi > \pi/4$. The value ϕ next to $\phi = \pi/4$ for which tables exist is $\phi = 5\pi/18 = 50^\circ$. Sample calculations are made for this value of ϕ.

Asymptotic behaviour

Knowledge of asymptotic behaviour of functions J_0 and Y_0 is required to select proper combination in the range $r > a$. Set

$$J_0(\lambda r) + iY_0(\lambda r) = H_0^{(1)}(\lambda r) = H_0^{(1)}(\rho r e^{i\phi}) \tag{11a}$$

Then

$$J_0(\lambda r) - iY_0(\lambda r) = H_0^{(2)}(\lambda r) = H_0^{(2)}(\rho r e^{i\phi}) \tag{11b}$$

$$H_0^{(1)}(\lambda r) \sim \frac{1}{\sqrt{\lambda r}} e^{i\lambda r} = \frac{1}{\sqrt{\lambda r}} e^{i\rho r[\cos\phi + i\sin\phi]} \tag{12a}$$

$$H_0^{(2)}(\lambda r) \sim \frac{1}{\sqrt{\lambda r}} e^{-i\lambda r} = \frac{1}{\sqrt{\lambda r}} e^{-i\rho r[\cos\phi + i\sin\phi]} \tag{12b}$$

Since $\sin\phi > 0$ we have that $H_0^{(1)}(\lambda r)$ decays exponentially for large r and $H_0^{(2)}(\lambda r)$ increases exponentially for large r. The reverse is true for $H_0^{(1)}(\bar{\lambda} r)$ and $H_0^{(2)}(\bar{\lambda} r)$. Accordingly, in the range $r > a$ the proper combinations of Bessel functions which must be used are $H_0^{(1)}(\lambda r)$ and $H_0^{(2)}(\bar{\lambda} r)$.

When $r < a$ the singular behaviour of the function Y_0 for $r = 0$ excludes them from the solution of the given problem.

General solution of differential equation (1)

In view of the above considerations the general solution of (1) is taken in the form

$$r \leqslant a: \quad w = (p_0/k)\left[1 + C_1 J_0(\lambda r) + \bar{C}_1 J_0(\bar{\lambda} r)\right] \tag{13a}$$

$$a < r: \quad w = (p_0/k)\left[C_2 H_0^{(1)}(\lambda r) + \bar{C}_2 H_0^{(2)}(\bar{\lambda} r)\right] \tag{13b}$$

Equations (13) are the natural extensions of equations (A9) and (A15), to which they reduce, for the case $\eta = 1$, when $N = 0$ and therewith $\phi = \pi/4$.

Evaluation of transition conditions

Form of transition conditions is

$$r = a: \quad w,\ w',\ \nabla^2 w,\ (\nabla^2 w)' \ \text{continuous} \tag{14}$$

Setting again

$$\lambda a = \mu \tag{A18}$$

and making use of the differentiation formulas (A16) and (A17) there are obtained the following transition equations:

$$1 + C_1 J_0(\mu) + \bar{C}_1 J_0(\bar{\mu}) = C_2 H_0^{(1)}(\mu) + \bar{C}_2 H_0^{(2)}(\bar{\mu}) \tag{14a}$$

$$C_1 \lambda J_1(\mu) + \bar{C}_1 \bar{\lambda} J_1(\bar{\mu}) = C_2 \lambda H_1^{(1)}(\mu) + \bar{C}_2 \bar{\lambda} H_1^{(2)}(\bar{\mu}) \tag{14b}$$

$$C_1 \lambda^2 J_0(\mu) + \bar{C}_1 \bar{\lambda}^2 J_0(\bar{\mu}) = C_2 \lambda^2 H_0^{(1)}(\mu) + \bar{C}_2 \bar{\lambda}^2 H_0^{(2)}(\bar{\mu}) \tag{14c}$$

$$C_1 \lambda^3 J_1(\mu) + \bar{C}_1 \bar{\lambda}^3 J_1(\bar{\mu}) = C_2 \lambda^3 H_1^{(1)}(\mu) + \bar{C}_2 \bar{\lambda}^3 H_1^{(2)}(\bar{\mu}) \tag{14d}$$

In order to simplify the system (14) we use (4) and write

$$\bar{\lambda}^2 = -\lambda^2 - N/D, \qquad \bar{\mu}^2 = -\mu^2 - aN/D \tag{15}$$

Therewith (14) may be brought into the following alternate form

$$\left.\begin{aligned} &C_2 H_0^{(1)}(\mu) + \bar{C}_2 H_0^{(2)}(\bar{\mu}) - C_1 J_0(\mu) - \bar{C}_1 J_0(\bar{\mu}) = 1 \\ &C_2 H_0^{(1)}(\mu) - \bar{C}_2 H_0^{(2)}(\bar{\mu}) - C_1 J_0(\mu) + \bar{C}_1 J_0(\bar{\mu}) \\ &\qquad - \frac{aN/D}{\mu^2}\left[\bar{C}_2 H_0^{(2)}(\mu) - \bar{C}_1 J_0(\bar{\mu})\right] = 0 \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} &C_2 \mu H_1^{(1)}(\mu) + \bar{C}_2 \bar{\mu} H_1^{(2)}(\bar{\mu}) - C_1 \mu J_1(\mu) - \bar{C}_1 \bar{\mu} J_1(\bar{\mu}) = 0 \\ &C_2 \mu H_1^{(1)}(\mu) - \bar{C}_2 \bar{\mu} H_1^{(2)}(\bar{\mu}) - C_1 \mu J_1(\mu) + \bar{C}_1 \bar{\mu} J_1(\bar{\mu}) \\ &\qquad - \frac{aN/D}{\mu^2}\left[\bar{C}_2 \bar{\mu} H_1^{(2)}(\bar{\mu}) - \bar{C}_1 \bar{\mu} J_1(\bar{\mu})\right] = 0 \end{aligned}\right\} \tag{17}$$

From (16), by subtraction,

$$\bar{C}_2 \left[2 + \frac{aN/D}{\bar{\mu}^2}\right] H_0^{(2)}(\bar{\mu}) - \bar{C}_1 \left[2 + \frac{aN/D}{\bar{\mu}^2}\right] J_0(\bar{\mu}) = 1 \qquad (18)$$

From (17), by subtraction,

$$\bar{C}_2 \bar{\mu} \left[2 + \frac{aN/D}{\bar{\mu}^2}\right] H_1^{(2)}(\bar{\mu}) - \bar{C}_1 \bar{\mu} \left[2 + \frac{aN/D}{\bar{\mu}^2}\right] J_1(\bar{\mu}) = 0 \qquad (19)$$

Equations (18) and (19) may be rewritten in simpler form. Taking conjugates of everything and observing that $\overline{H_1^{(2)}(\bar{\mu})} = H_1^{(1)}(\mu)$

$$\begin{aligned} C_2 H_0^{(1)}(\mu) - C_1 J_0(\mu) &= \frac{1}{2 + aN/D\,\bar{\mu}^2} \\ C_2 H_1^{(1)}(\mu) - \bar{C}_1 J_1(\mu) &= 0 \end{aligned} \qquad (20)$$

Therewith

$$\begin{aligned} C_1 &= \frac{1}{2 + N/D\,\bar{\lambda}^2} \; \frac{-H_1^{(1)}(\mu)}{\Delta} \\ C_2 &= \frac{1}{2 + N/D\,\bar{\lambda}^2} \; \frac{-J_1(\mu)}{\Delta} \end{aligned} \qquad (21)$$

where

$$\begin{aligned} \Delta &= -H_0^{(1)}(\mu) J_1(\mu) + H_1^{(1)}(\mu) J_0(\mu) \\ &= -(J_0 + iY_0)J_1 + (J_1 + iY_1) = i(Y_0 J_1 - Y_1 J_0) = \frac{-2i}{\pi\mu} \end{aligned} \qquad (22)$$

Further transformations are carried out as follows, with the help of (5) to (8)

$$\frac{1}{2 + N/D\,\bar{\lambda}^2} = \frac{1}{2 - 2\rho^2\cos 2\phi/\rho^2 e^{-2i\phi}} = \frac{1}{2 - 2\cos 2\phi\, e^{2i\phi}}$$

$$= \frac{1}{2 - 2\cos^2 2\phi - 2i\cos 2\phi \sin 2\phi}$$

$$= \frac{1}{2\sin^2 2\phi - 2i\cos 2\phi \sin 2\phi}$$

$$= \frac{\sin 2\phi + i\cos 2\phi}{2\sin 2\phi} = \frac{1}{2}(1 + i\cot 2\phi) \qquad (23)$$

Introducing (22) and (23) into (21) gives

$$\begin{aligned} C_1 &= \frac{\pi\mu}{4i}\left[1 + i\cot 2\phi\right] H_1^{(1)}(\mu) \\ C_2 &= \frac{\pi\mu}{4i}\left[1 + i\cot 2\phi\right] J_1(\mu) \end{aligned} \qquad (24)$$

We transform C_1 further,

$$C_1 = \frac{\pi \rho a}{4i} e^{i\phi}(1 + i\cot 2\phi)\left[J_1 + iY_1\right]$$

$$= \frac{1}{4}\pi \rho a(\sin\phi - i\cos\phi)(1 + i\cot 2\phi)(J_1 + iY_1)$$

$$= \frac{1}{4}\pi \rho a\left[\sin\phi + \cos\phi\cot 2\phi + i(\sin\phi\cot 2\phi - \cos\phi)\right](J_1 + iY_1)$$

$$= \frac{1}{4}\pi \rho a \; \frac{\cos\phi - i\sin\phi}{\sin 2\phi}\left[(u_1 - V_1) + i(v_1 + U_1)\right]$$

$$C_1 = \frac{\pi \rho a}{4\sin 2\phi}\Big[\cos\phi(u_1 - V_1) + \sin\phi(v_1 + U_1)$$

$$+ i\left\{\cos\phi(v_1 + U_1) - \sin\phi(u_1 - V_1)\right\}\Big] \qquad (25)$$

Expression for foundation pressure at center

From equation (13a)

$$\frac{\sigma_F(0)}{p_0} = \frac{kw(0)}{p_0} = 1 + C_1 + \overline{C}_1 \qquad (26)$$

Combination of (26) and (25) gives the following explicit formula

$$\frac{\sigma_F(0)}{p_0} = 1 + \frac{\pi \rho a}{4}\left[\frac{u_1(\rho a,\phi) - V_1(\rho a,\phi)}{\sin\phi}\right.$$

$$\left. + \frac{v_1(\rho a,\phi) + U_1(\rho a,\phi)}{\cos\phi}\right] \qquad (27)$$

which is equation (16) of the main body of the report.

Expression for bending stress at center

Analogy between (13a) and equation (A9) indicates that the general formula (A35), with $\eta = 1$, remains applicable for the case of the plate under tension. Accordingly,

$$\frac{\sigma_B(0)}{p_0} = \mp \frac{1 + \nu}{4} \frac{Eh}{(1 - \nu^2)} \frac{1}{ka^2}\left[\mu^2 C_1 + \bar{\mu}^2 \overline{C}_1\right] \qquad (28)$$

In this, in view of (24),

$$\mu^2 C_1 = \frac{\pi(\rho a)^3}{4i} e^{i3\phi}(1 + i\cot 2\phi)(J_1 + iY_1)$$

$$= \frac{\pi(\rho a)^3}{4i}\Big[(\cos 3\phi - \sin 3\phi\cot 2\phi)$$

$$+ i(\sin 3\phi + \cos 3\phi\cot 2\phi)\Big](J_1 + iY_1)$$

$$= \frac{\pi(\rho a)^3}{4i}\left[\frac{-\sin\phi + i\cos\phi}{\sin 2\phi}\right](J_1 + iY_1)$$

$$= \frac{\pi(\rho a)^3}{8i}\left(\frac{1}{\sin\phi} - \frac{1}{\cos\phi}\right)(J_1 + iY_1)$$

$$\mu^2 c_1 = \frac{\pi(\rho a)^3}{8}\left[\frac{u_1 - V_1}{\sin\phi} - \frac{v_1 + U_1}{\cos\phi} + i\left\{\cdots\cdot\right\}\right] \tag{25}$$

Combination of (28) and (29) gives the following formula

$$\frac{\sigma_B(0)}{p_0} = \mp \frac{Eh}{1-\nu}\,\frac{(\rho a)^3\pi}{16ka^2}\left[\frac{u_1 - V_1}{\sin\phi} - \frac{v_1 + U_1}{\cos\phi}\right] \tag{30}$$

Equation (30) may be written in the alternate form

$$\frac{\sigma_B(0)}{p_0} = \mp\sqrt{\frac{3(1+\nu)}{4(1-\nu)}}\,\sqrt{\frac{E}{kh}}\,\frac{\pi}{4}\,\rho a\left[\frac{u_1 - V_1}{\sin\phi} - \frac{v_1 + U_1}{\cos\phi}\right] \tag{31}$$

and this formula is equation (17) of the main body of this report.

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1955

PROCEEDINGS OF THE SOCIETY